THE STUDY OF A CRITICAL POINT IN AN EQUATION OF TRANSFER THEORY

R. S. Levitin

UDC 536.24:517.9

An investigation was carried out to establish the conditions for the appearance and disappearance of a compressed bundle of trajectories, leaving from a critical point of an equation in the self-similar case under the presence of nonsmooth perturbing forces.

1°. Some problems of transfer theory (heat-transfer diffusion) in the presence of self-similarity are reduced to ordinary differential equations of the form

$$\alpha(x)\frac{dz}{dx} = F(x, z), \tag{1}$$

for which it is necessary to elucidate the existence and the behavior of O-curves near the origin (0, 0) which is a singular point. First we give the following theorem, from which follows the formulation of the problem.

The Hartman–Wintner Theorem [1]. Assume that in Eq. (1):

1) the function F(x, z) is defined and continuous with respect to x, z in the domain

$$0 < x \leqslant x_0, \quad -z_0 \leqslant z \leqslant z_0, \tag{2}$$

where x_0 and z_0 are positive numbers; $1/F(x, z)$ is bounded for $0 < x \leq \varepsilon$, $\tau \leq |z| \leq z_0$, where $\tau > 0$ is an arbitrarily small; ε is a sufficiently small number;

2) $F(x, -z_0) > 0$, $F(x, z_0) < 0$ for $0 < x \leq x_0$;

3) the function $\alpha(x)$ is defined, continuous and positive for $0 < x \leq x_0$,

$$\int_0^{x_0} \frac{dx}{\alpha(x)} = +\infty.$$

Then there exists at least one solution $z = z(x)$ of Eq. (1), which is defined in the interval $0 < x \leq x_0$, and every such solution has the property: $z(x) \to 0$ for $x \to 0$ (i.e., represents an O-curve of Eq. (1)).

Thus, under the conditions of the above given theorem for Eq. (1), there exists in the domain (2) either one O-curve or an uncountable set of O-curves, i.e., there arises the problem of distinguishing the O-curves.

Below we give uniqueness and nonuniqueness theorems for the O-curves.

In [2], Andreev establishes a uniqueness theorem for O-curves if the conditions of the Hartman–Wintner theorem are satisfied. To this end he gives the following lemma.

LEMMA. We assume that for Eq. (1):

1) the conditions of the Hartman–Wintner theorem hold;

2) in the domain (2) (or any subdomain of the same form) for

$$F(x, z_1) - F(x, z_2) \leqslant \lambda(x)(z_1 - z_2),$$

Institute of Heat and Mass Transfer, Academy of Sciences of the Belorussian SSR, Minsk. Translated from Inzhenerno-Fizicheskii Zhurnal, Vol. 17, No. 3, pp. 565–569, September, 1969. Original article submitted October 25, 1968.

where $\lambda(x)$ is continuous for $x \in (0, x_0]$, we have

$$\int_x^{x_0} \frac{\lambda(x)}{\alpha(x)} dx \leqslant M < +\infty, \tag{3}$$

M being a constant and x any number from $(0, x_0)$.

Then, Eq. (1) has a unique O-curve in the domain (2).

This lemma is an immediate generalization of Lohn's lemma ([3], p. 115) and can be proved by the same simple method.

In order to obtain a stronger statement about the uniqueness of the O-curve, Andreev makes, naturally, additional assumptions relative to the structure of the right-hand side of (1) (for example, its representability in the form $F(x, z) = \Phi(x, z) + \eta(x, z)$ and the Lipschitz condition with respect to z). His uniqueness theorem establishes a relation between the Lipschitz coefficient, depending on x, the comparison introduced by him, which characterizes the smallness of the perturbation $\eta(x, z)$, and the function $\alpha(x)$.

If we seek the conditions for the uniqueness of the O-curve for (1) in the case when $F(x, z)$ is from the Osgood class, which will be defined below in a precise manner, then, apparently, the following theorem which generalizes Lohn's lemma will be useful.

THEOREM 1. Assume that for Eq. (1):

1) the Hartman–Wintner conditions hold;

2) in the domain (2) (or any subdomain of the same form) for $z_1 > z_2$ we have

$$F(x, z_1) - F(x, z_2) \leqslant \lambda(x)\, \omega_3(z_1 - z_2),$$

where

$$\int_x^{x_0} \frac{\lambda(x)}{\alpha(x)} dx \leqslant M < +\infty,$$

$\omega_3(\tau)$ is a continuous function, $\omega_3(0) = 0$, $\omega_3(\tau) > 0$ for $\tau > 0$,

$$\int_0^{\nu} \frac{d\tau}{\omega_3(\tau)} < \infty \quad (\nu > 0).$$

Then Eq. (1) has in the domain (2) a unique O-curve.

Hartman and Wintner apply their theorem to the generalized Briot–Bouquet equation

$$x^q z' = -pz + f(x, z), \tag{4}$$

where $q \geq 1$; $f(x, z)$ is a real single-valued function, continuous in the rectangle

$$0 \leqslant x \leqslant x_0, \quad -z_0 \leqslant z \leqslant z_0;$$

finally, p is an arbitrary positive number.

They have obtained the following corollary.

COROLLARY. If

$$f(0, z) = o(z) \quad \text{for} \quad z \to 0$$

(in fact it is sufficient if $|f(0, z)| \leq p|z|$ for small $|z|$, $0 < |z| \leq z_0$), then Eq. (4) has at least one solution $z = z(x)$, which exists for all small $x > 0$, and all these solutions are O-curves.

Then, the authors of [1] make the following remark: if the conditions of the corollary hold, then in order to guarantee the uniqueness of the O-curve it is sufficient to require the growth of $f(x, z)$ with z.

For our part we obtain the following corollary.

COROLLARY. For the uniqueness of the O-curve it is sufficient to require that

$$f(x, z_1) - f(x, z_2) \leqslant \lambda(x)\,\omega_3(z_1 - z_2) \quad (z_1 > z_2),$$

where $\lambda(x)$ and $\omega_3(x)$ are defined as in Theorem 1.

2°. We consider the "algebroid" differential equation

$$y' = \frac{P_n(x, y) + f_n(x, y)}{Q_n(x, y) + g_n(x, y)}, \tag{5}$$

where P_n, Q_n are homogeneous polynomials of integer exponent $n > 1$:

$$P_n = a_{n-1} y x^{n-1} + \sum_{k=2}^{n} a_{n-k} y^k x^{n-k}, \tag{6}$$

$$Q_n = b_n x^n + \sum_{k=2}^{n} b_{n-k} y^k x^{n-k}, \tag{7}$$

while $f_n(x, y)$, $g_n(x, y)$ are continuous in the domain $R_1: |x| \leq a$, $|y| \leq b$ and

$$f_n,\ g_n = o(r^n) \quad \text{for} \quad r = |x| + |y| \to 0. \tag{8}$$

In the formulas (6), (7) in the case $n = 1$, the empty sums are considered to be equal to zero.

We assume that

$$(a_{n-1} - b_n)\, b_n < 0.$$

Then for (5) there arises Frommer's first discernment problem (see [3], p. 108: $G(0) = b_n$, $C = a_{n-1} - b_n$, i.e., $CG(0) = (a_{n-1} - b_n)b_n < 0$), i.e., it is necessary to clarify whether one or several integral curves (K-curves) tend to the point (0, 0) along simple exceptional directions which are semiaxes of the x-axis.

A survey and criticism of the papers dealing with the indicated problem is given by Andreev [4].

We give a nonuniqueness theorem and a uniqueness theorem for the perturbations f_n, g_n from the Osgood and Tamarkine classes which are defined below. We note that, as a rule, the authors usually assume that f_n and g_n are from the Lipschitz class.

For our purposes it is sufficient to consider the domain

$$R: 0 \leqslant x \leqslant a, \quad |y| \leqslant b \tag{9}$$

and, without loss of generality, to assume

$$\mu < b_n = 1, \quad \mu \overset{df}{=} a_{n-1}.$$

Definition 1. We consider the family $\{f(x, y)\}$ of functions, Lebesgue measurable with respect to x for each fixed y and continuous with respect to y for each fixed x from the domain R. This family will be called a Tamarkine class if there exist measurable nonnegative functions $s(x)$ and $\omega(t)$, $s(x) > 0$ for $x > 0$, $\omega(t) > 0$ for $t > 0$,

$$\int_0^t \frac{dy}{\omega(y)} < \infty \quad (t > 0),$$

such that in R we have

$$|f(x, y) - f(x, y(x))| \geqslant s(x)\,\omega(|y - y(x)|),^*$$

where $y = y(x)$ is some continuous curve from the domain R.

The family $\{f(x, y)\}$ will be called an Osgood class if there exist measurable nonnegative functions $s_1(x)$ and $\omega_1(t)$

---

* For some subclass of such functions $f \in C$, J. Tamarkine [5] has proved a nonuniqueness theorem for the solution of the Cauchy problem of $y' = f(x, y)$.

$$\int_0^t \frac{dt}{\omega_1(t)} = +\infty^* \qquad (t > 0), \tag{10}$$

such that in R we have

$$|f(x, y_1) - f(x, y_2)| \leqslant s_1(x)\,\omega_1(|y_1 - y_2|).$$

Definition 2. A continuous curve $y = y(x)$ from the domain R will be called a K-curve if $y(x)/x \to 0$ for $x \to 0$, and will be called a K-solution if it also satisfies (5).

Two K-solutions $y = y_1(x)$ and $y = y_2(x)$ are said to be essentially distinct, if $\exists x_0 \in (0, a']$ such that $\forall x \in (0, x_0)\; y_1(x) \neq y_2(x)$.

The K-solutions $y = Y_1(x)$ and $y = Y_2(x)$ will be called respectively the upper and the lower K-solutions of (5), if for every K-solution $y = y(x)$ of (5) we have: $Y_2(x) \leq y(x) \leq Y_1(x)$. It is easy to prove the existence and the uniqueness of such solutions.

We consider the class of pairs of functions f_n, g_n, formed by functions f_n from the Tamarkine class and by functions g_n from the Osgood class, such that the following conditions hold in R. For f_n:

$$|f_n(x, y) - f_n(x, \theta(x))| \geqslant s(x)\,x^{n-1}\omega(|y - \theta(x)|), \tag{11}$$

where $y = \theta(x)$ is some fixed K-solution of (5); $s(x)$ and $\omega(t)$ are measurable and nonnegative; $s(x) > 0$ for $x > 0$, $\omega(t) > 0$ for $t > 0$,

$$\int_0^\nu \frac{dt}{\omega(t)} < \infty \qquad (\nu > 0), \tag{12}$$

and for g_n at least one of the following two inequalities holds

$$|g_n(x, y_1) - g_n(x, y_2)| \leqslant L x^{n-1} |y_1 - y_2|, \;(L \geqslant 0), \tag{13}$$

$$|g_n(x, y_1) - g_n(x, y_2)| \leqslant s(x)\,x^{n-1}\omega(|y_1 - y_2|).\dagger \tag{14}$$

We denote

$$h_n = f_n(x, y) - f_n(x, \theta(x)).$$

It follows from (11) that we can have only the following possibilities for the sign of h_n in $R \setminus \{x, \theta(x)\}$ for $0 < x \leq a$:

a) h_n preserves the sign; b) $h_n > 0$ for $y > \theta(x)$ and $h_n < 0$ for $y < \theta(x)$; c) $h_n < 0$ for $y > \theta(x)$ and $h_n > 0$ for $y < \theta(x)$.

For the pair of functions mentioned we have the following theorem.

THEOREM 2. Equation (5) for $0 < \mu < 1$ has at least two essentially distinct K-solutions (hence, infinitely many) in the cases a) and b). In the case c), $y = \theta(x)$ is the unique K-solution. In the case a) for $h > 0$ $(h < 0)$ $Y_2(x) \equiv \theta(x)$ $(Y_1(x) \equiv \theta(x))$, and in the case b) $Y_2(x) < \theta(x) < Y_1(x)$ for $x > 0$.

Now we give a uniqueness theorem for the K-solutions.

THEOREM 3. Assume that for Eq. (5) we have the domain R

$$|f_n(x, y_1) - f_n(x, y_2)| \leqslant H(x)\,x^{n-1}\omega(|y_1 - y_2|),$$

where $H(x)$ is continuous for $x > 0$ and

$$\int_0^\nu \frac{H(x)}{x}\,dx < \infty, \tag{15}$$

* Naturally, condition (10) is assumed only in the case when the function $\omega_1(t)$ is not equivalent to zero.
† In the inequality (14) the functions $s(x)$ and $\omega(x)$ are the same as in (11).

$\omega(t)$ is continuous for $t \geq 0$, $\omega(0) = 0$, $\omega(t) > 0$ for $t > 0$ and

$$\int_0^\nu \frac{dt}{\omega(t)} = +\infty \quad (\nu > 0),$$

and for g_n at least one of the following two conditions holds: 1) (13) is satisfied; 2) g_n is from the same class as f_n. Then (5) has a unique K-solution.

We note that one cannot replace condition (15) by one of the two conditions

$$1) \quad \int_0^\nu \frac{H(x)}{x} dx = \infty \quad (\nu > 0);$$

$$2) \quad \int_0^\nu \frac{H(x)}{x^{1-\delta}} dx < \infty \quad (\delta > 0 \text{ is arbitrary}),$$

since a bundle of distinct K-solutions can appear.

Theorems 2 and 3 for Osgood and Tamarkine classes form a result close to the criterion.

An important fact recently established experimentally is the discontinuance of the diffusion in the critical domain of stratification [6]. By this, the nonlinear character of the diffusion equation is established. In the determination of the self-similar solutions of the nonlinear parabolic equation of the isothermal diffusion, there arises the problem of distinguishing the O-curves.

LITERATURE CITED

1. P. Hartman and A. Wintner, Amer. J. Math., 68, No. 2, 301-308 (1946).
2. A. F. Andreev, Differents. Uravnen., No. 1 (1965).
3. V. V. Nemytskii and V. V. Stepanov, Qualitative Theory of Differential Equations [in Russian], Moscow (1949).
4. A. F. Andreev, Proc. of the Fourth All-Union Mathematical Congress [in Russian], Vol. 2, Leningrad (1964).
5. J. Tamarkine, Math. Zeitschr., 16, 207-213 (1923).
6. I. R. Krichevskii and Yu. V. Tsekhanskaya, Zh. Fiz. Khim., 30, 2315 (1956).